

On a theorem of V. Bernik in the metrical theory of Diophantine approximation

by

V. BERESNEVICH (MINSK)¹

1. Introduction. We begin by introducing some notation: $\#S$ will denote the number of elements in a finite set S ; the Lebesgue measure of a measurable set $S \subset \mathbb{R}$ will be denoted by $|S|$; P_n will be the set of integral polynomials of degree $\leq n$. Given a polynomial P , $H(P)$ will denote the height of P , *i.e.* the maximum of the absolute values of its coefficients; $P_n(H) = \{P \in P_n : H(P) = H\}$. The symbol of Vinogradov \ll in the expression $A \ll B$ means $A \leq CB$, where C is a constant. The symbol \asymp means both \ll and \gg . Given a point $x \in \mathbb{R}$ and a set $S \subset \mathbb{R}$, $\text{dist}(x, S) = \inf\{|x - s| : x \in S\}$. Throughout, Ψ will be a positive function.

Mahler's problem. In 1932 K. Mahler [10] introduced a classification of real numbers x into the so-called classes of A, S, T and U numbers according to the behavior of $w_n(x)$ defined as the supremum of $w > 0$ for which

$$|P(x)| < H(P)^{-w}$$

holds for infinitely many $P \in P_n$. By Minkowski's theorem on linear forms, one readily shows that $w_n(x) \geq n$ for all $x \in \mathbb{R}$. Mahler [9] proved that for almost all $x \in \mathbb{R}$ (in the sense of Lebesgue measure) $w_n(x) \leq 4n$, thus almost all $x \in \mathbb{R}$ are in the S -class. Mahler has also conjectured that for almost all $x \in \mathbb{R}$ one has the equality $w_n(x) = n$. For about 30 years the progress in Mahler's problem was limited to $n = 2$ and 3 and to partial results for $n > 3$. V. Sprindzuk proved Mahler's conjecture in full (see [12]).

A. Baker's conjecture. Let $W_n(\Psi)$ be the set of $x \in \mathbb{R}$ such that there are infinitely many $P \in P_n$ satisfying

$$(1) \quad |P(x)| < \Psi(H(P)).$$

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A. Baker [1] has improved Sprindžuk's theorem by showing that

$$|W_n(\Psi)| = 0 \text{ if } \sum_{h=1}^{\infty} \Psi^{1/n}(h) < \infty \text{ and } \Psi \text{ is monotonic.}$$

He has also conjectured a stronger statement proved by V. Bernik [4] that $|W_n(\Psi)| = 0$ if the sum

$$(2) \quad \sum_{h=1}^{\infty} h^{n-1} \Psi(h)$$

converges and Ψ is monotonic. Later V. Beresnevich [5] has shown that $|\mathbb{R} \setminus W_n(\Psi)| = 0$ if (2) diverges and Ψ is monotonic. We prove

THEOREM 1. *Let $\Psi : \mathbb{R} \rightarrow \mathbb{R}^+$ be arbitrary function (not necessarily monotonic) such that the sum (2) converges. Then $|W_n(\Psi)| = 0$.*

Theorem 1 is no longer improvable as, by [5], the convergence of (2) is crucial. Notice that for $n = 1$ the theorem is simple and known (see, for example, [8, p. 121]). Therefore, from now on we assume that $n \geq 2$.

2. Subcases of Theorem 1. Let $\delta > 0$. We define the following 3 sets denoted by $W_{\text{big}}(\Psi)$, $W_{\text{med}}(\Psi)$ and $W_{\text{small}}(\Psi)$ consisting of $x \in \mathbb{R}$ such that there are infinitely many $P \in P_n$ simultaneously satisfying (1) and one of the following inequalities

$$(3) \quad 1 \leq |P'(x)|,$$

$$(4) \quad H(P)^{-\delta} \leq |P'(x)| < 1,$$

$$(5) \quad |P'(x)| < H(P)^{-\delta}$$

respectively. Obviously $W_n(\Psi) = W_{\text{big}}(\Psi) \cup W_{\text{med}}(\Psi) \cup W_{\text{small}}(\Psi)$. Hence to prove Theorem 1 it suffices to show that each of the sets has zero measure.

Since sum (2) converges, $H^{n-1}\Psi(H)$ tends to 0 as $H \rightarrow \infty$. Therefore,

$$(6) \quad \Psi(H) = o(H^{-n+1}) \text{ as } H \rightarrow \infty.$$

3. The case of a big derivative. The aim of this section is to prove that $|W_{\text{big}}(\Psi)| = 0$. Let $B_n(H)$ be the set of $x \in \mathbb{R}$ such that there exists a polynomial $P \in P_n(H)$ satisfying (3). Then

$$(7) \quad W_{\text{big}}(\Psi) = \bigcap_{N=1}^{\infty} \bigcup_{H=N}^{\infty} B_n(H).$$

Now $|W_{\text{big}}(\Psi)| = 0$ if $|W_{\text{big}}(\Psi) \cap I| = 0$ for any open interval $I \subset \mathbb{R}$ satisfying

$$(8) \quad 0 < c_0(I) = \inf\{|x| : x \in I\} < \sup\{|x| : x \in I\} = c_1(I) < \infty.$$

Therefore we can fix an interval I satisfying (8).

By (7) and the Borel-Cantelli Lemma, $|W_{\text{big}}(\Psi) \cap I| = 0$ whenever

$$(9) \quad \sum_{H=1}^{\infty} |B_n(H) \cap I| < \infty.$$

By the convergence of (2), condition (9) will follow on showing that

$$(10) \quad |B_n(H) \cap I| \ll H^{n-1} \Psi(H)$$

with the implicit constant in (10) independent of H .

Given a $P \in P_n(H)$, let $\sigma(P)$ be the set of $x \in I$ satisfying (3). Then

$$(11) \quad B_n(H) \cap I = \bigcup_{P \in P_n(H)} \sigma(P).$$

Lemma 1. *Let I be an interval with endpoints a and b . Define the following sets $I'' = [a, a + 4\Psi(H)] \cup [b - 4\Psi(H), b]$ and $I' = I \setminus I''$. Then for all sufficiently large H for any $P \in P_n(H)$ such that $\sigma(P) \cap I' \neq \emptyset$, for any $x_0 \in \sigma(P) \cap I'$ there exists $\alpha \in I$ such that $P(\alpha) = 0$, $|P'(\alpha)| > |P'(x_0)|/2$ and $|x_0 - \alpha| < 2\Psi(H) |P'(\alpha)|^{-1}$.*

The proof of this Lemma nearly coincides with the one of Lemma 1 in [5] and is left for the reader. There will be some changes to constants and notation and one also will have to use (6).

Given a polynomial $P \in P_n(H)$ and a real number α such that $P'(\alpha) \neq 0$, define $\sigma(P; \alpha) = \{x \in I : |x - \alpha| < 2\Psi(H) |P'(\alpha)|^{-1}\}$. Let I' and I'' be defined as in Lemma 1. For every polynomial $P \in P_n(H)$, we define the set

$$Z_I(P) = \{\alpha \in I : P(\alpha) = 0 \text{ and } |P'(\alpha)| \geq 1/2\}.$$

By Lemma 1, for any $P \in P_n(H)$ we have the inclusion

$$(12) \quad \sigma(P) \cap I' \subset \bigcup_{\alpha \in Z_I(P)} \sigma(P; \alpha).$$

Given $k \in \mathbb{Z}$ with $0 \leq k \leq n$, define

$$P_n(H, k) = \{P = a_n x^n + \cdots + a_0 \in P_n(H) : a_k = 0\}$$

and for $R \in P_n(H, k)$ let $P_n(H, k, R) = \{P \in P_n(H) : P - R = a_k x^k\}$. It is easily observed that

$$(13) \quad P_n(H) = \bigcup_{k=0}^n \bigcup_{R \in P_n(H, k)} P_n(H, k, R)$$

and

$$(14) \quad \#P_n(H, k) \ll H^{n-1} \quad \text{for every } k.$$

Taking into account (11), (13), (14) and that $|I''| \ll \Psi(H)$, it now becomes clear that to prove (10) it is sufficient to show that for every fixed k and fixed $R \in P_n(H, k)$

$$(15) \quad \left| \bigcup_{P \in P_n(H, k, R)} \sigma(P) \cap I' \right| \ll \Psi(H).$$

Let k and R be fixed. Define the rational function $\tilde{R}(x) = x^{-k}R(x)$. By (8), there exists a collection of intervals $[w_{i-1}, w_i] \subset I$ ($i = 1, \dots, s$), which do not intersect pairwise and cover I , such that $\tilde{R}(x)'$ is monotonic and does not change the sign on every interval $[w_{i-1}, w_i]$. It is clear that s depends on n only. Let $Z_{I,R} = \bigcup_{P \in P_n(H, k, R)} Z_I(P)$, $k_i = \#(Z_{I,R} \cap [w_{i-1}, w_i])$ and $Z_{I,R} \cap [w_{i-1}, w_i] = \{\alpha_i^{(1)}, \dots, \alpha_i^{(k_i)}\}$, where $\alpha_i^{(j)} < \alpha_i^{(j+1)}$. Given a $P \in P_n(H, k, R)$, we obviously have the identity

$$\frac{x^k P'(x) - kx^{k-1}P(x)}{x^{2k}} = \left(\frac{P(x)}{x^k} \right)' = \tilde{R}(x)'.$$

Taking x to be $\alpha \in Z_I(P)$ leads to $\frac{P'(\alpha)}{\alpha^k} = \tilde{R}(\alpha)'$. By (8), $|P'(\alpha)| \asymp |\tilde{R}(\alpha)'|$. Now, by Lemma 1, $|\sigma(P; \alpha)| \ll \Psi(H) |P'(\alpha)|^{-1} \ll \Psi(H) |\tilde{R}'(\alpha)|^{-1}$.

Using (12), we get

$$\left| \bigcup_{P \in P_n(H, k, R)} \sigma(P) \cap I' \right| \ll \Psi(H) \sum_{i=1}^s \sum_{j=1}^{k_i} \frac{1}{|\tilde{R}(\alpha_i^{(j)})|}$$

Now to show (15) it suffices to prove that for every i ($1 \leq i \leq s$)

$$(16) \quad \sum_{j=1}^{k_i} |\tilde{R}'(\alpha_i^{(j)})|^{-1} \ll 1.$$

Fix an index i ($1 \leq i \leq s$). If $k_i \geq 2$ then we can consider two sequential roots $\alpha_i^{(j)}$ and $\alpha_i^{(j+1)}$ of two rational functions $\tilde{R} + a_k^{i,j}$ and $\tilde{R} + a_k^{i,j+1}$ respectively. For convenience let us assume that \tilde{R}' is increasing and positive on $[w_{i-1}, w_i)$. Then \tilde{R} is strictly monotonic on $[w_{i-1}, w_i)$, and we have $a_k^{i,j} \neq a_k^{i,j+1}$. It follows that $|a_k^{i,j} - a_k^{i,j+1}| \geq 1$. Using The Mean Value Theorem and the monotonicity of \tilde{R}' , we get

$$1 \leq |a_0^{i,j} - a_0^{i,j+1}| = |\tilde{R}'(\alpha_i^{(j)}) - \tilde{R}'(\alpha_i^{(j+1)})| =$$

$$= |\tilde{R}'(\tilde{\alpha}_i^{(j)})| \cdot |\alpha_i^{(j)} - \alpha_i^{(j+1)}| \leq |\tilde{R}'(\alpha_i^{(j+1)})| \cdot |\alpha_i^{(j)} - \alpha_i^{(j+1)}|,$$

where $\tilde{\alpha}_i^{(j)}$ is a point between $\alpha_i^{(j)}$ and $\alpha_i^{(j+1)}$. This implies $|\tilde{R}'(\alpha_i^{(j+1)})|^{-1} \leq |\alpha_i^{(j+1)} - \alpha_i^{(j)}|$, whence we readily get

$$\sum_{j=1}^{k_i-1} |\tilde{R}'(\alpha_i^{(j+1)})|^{-1} \leq \sum_{j=1}^{k_i-1} (\alpha_i^{(j+1)} - \alpha_i^{(j)}) = \alpha_i^{(k_i)} - \alpha_i^{(1)} \leq w_i - w_{i-1}.$$

The last inequality and $|\tilde{R}'(\alpha_i^{(1)})| \asymp |P'(\alpha_i^{(1)})| \gg 1$ yield (16). It is easily verified that (16) holds for every i with $k_i \geq 2$ and is certainly true when $k_i = 1$ or $k_i = 0$. This completes the proof of the case of a big derivative.

4. The case of a medium derivative. As above we fix an interval I satisfying (8). The statement $|W_{\text{med}}(\Psi)| = 0$ will now follow from $|W_{\text{med}}(\Psi) \cap I| = 0$. We will use the following

Lemma 2 (see Lemma 2 in [6]). *Let $\alpha_0, \dots, \alpha_{k-1}, \beta_1, \dots, \beta_k \in \mathbb{R} \cup \{+\infty\}$ be such that $\alpha_0 > 0$, $\alpha_j > \beta_j \geq 0$ for $j = 1, \dots, k-1$ and $0 < \beta_k < +\infty$.*

Let $f : (a, b) \rightarrow \mathbb{R}$ be a $C^{(k)}$ function such that $\inf_{x \in (a, b)} |f^{(k)}(x)| \geq \beta_k$. Then, the set of $x \in (a, b)$ satisfying

$$\begin{cases} |f(x)| \leq \alpha_0, \\ \beta_j \leq |f^{(j)}(x)| \leq \alpha_j \quad (j = 1, \dots, k-1) \end{cases}$$

is a union of at most $k(k+1)/2 + 1$ intervals with lengths at most $\min_{0 \leq i < j \leq k} 3^{(j-i+1)/2} (\alpha_i / \beta_j)^{1/(j-i)}$. Here we adopt $\frac{c}{0} = +\infty$ for $c > 0$.

Given a polynomial $P \in P_n(H)$, we redefine $\sigma(P)$ to be the set of solutions of (4). Since $P^{(n)}(x) = n!a_n$, we can apply Lemma 2 to P with $k = n$ and

$$\begin{aligned} \alpha_0 &= \Psi(H), \quad \alpha_1 = 1, \quad \beta_1 = \inf_{x \in \sigma(P)} |P'(x)| \geq H^{-\delta}, \quad \beta_n = 1, \\ \alpha_2 &= \dots = \alpha_{n-1} = +\infty, \quad \beta_2 = \dots = \beta_{n-1} = 0. \end{aligned}$$

Then we conclude that $\sigma(P)$ is a union of at most $n(n+1)/2 + 1$ intervals of length $\ll \alpha_0/\beta_1$. There is no loss of generality in assuming that the sets $\sigma(P)$ are intervals as, otherwise, we would treat the intervals of $\sigma(P)$ separately. We also can ignore those P for which $\sigma(P)$ is empty. For every P we also define a point $\gamma_P \in \sigma(P)$ such that $\inf_{x \in \sigma(P)} |P'(x)| \geq \frac{1}{2}|P'(\gamma_P)|$. The existence is easily seen. Now we have

$$(17) \quad |\sigma(P)| \ll \Psi(H) |P'(\gamma_P)|^{-1}.$$

It also follows from the choice of γ_P that

$$(18) \quad H(P)^{-\delta} \leq |P'(\gamma_P)| < 1.$$

Now define expansions of $\sigma(P)$ as follows:

$$\begin{aligned} \sigma_1(P) &:= \{x \in I : \text{dist}(x, \sigma(P)) < (H|P'(\gamma_P)|)^{-1}\}, \\ \sigma_2(P) &:= \{x \in I : \text{dist}(x, \sigma(P)) < H^{-1+2\delta}\}. \end{aligned}$$

By (4), $\sigma_1(P) \subset \sigma_2(P)$. Moreover, it is easy to see that

$$(19) \quad \sigma_1(P) \subset \sigma_2(Q) \quad \text{for any } Q \in P_n(H) \text{ with } \sigma_1(Q) \cap \sigma_1(P) \neq \emptyset.$$

It is readily verified that $|\sigma_1(P)| \asymp (H|P'(\gamma_P)|)^{-1}$, and therefore, by (17),

$$|\sigma(P)| \ll |\sigma_1(P)| H \Psi(H).$$

Take any $x \in \sigma_2(P)$. Using the Mean Value Theorem, (18) and $|x - \gamma_P| \ll H^{-1+2\delta}$, we get $|P'(x)| \leq |P'(\gamma_P)| + |P''(\tilde{x})|(x - \gamma_P)| \ll 1 + H \cdot H^{-1+2\delta} \ll H^{2\delta}$, where \tilde{x} is between x and γ_P . Similarly we estimate $|P(x)|$ resulting in

$$(20) \quad |P(x)| \ll H^{-1+4\delta}, \quad |P'(x)| \ll H^{2\delta} \quad \text{for any } x \in \sigma_2(P).$$

Now for every pair (k, m) of integers with $0 \leq k < m \leq n$ we define

$$P_n(H, k, m) = \{R = a_n x^n + \cdots + a_0 \in P_n(H) : a_k = a_m = 0\}$$

and for a given polynomial $R \in P_n(H, k, m)$ we define

$$P_n(H, k, m, R) = \{P = R + a_m x^m + a_k x^k \in P_n(H)\}.$$

The intervals $\sigma(P)$ will be divided into 2 classes of essential and non-essential intervals. The interval $\sigma(P)$ will be essential if for any choice of (k, m, R) such that $P \in P_n(H, k, m, R)$ for any $Q \in P_n(H, k, m, R)$ other than P we have $\sigma_1(P) \cap \sigma_1(Q) = \emptyset$. For fixed k, m and R summing the measures of essential intervals gives

$$\sum |\sigma(P)| \leq H\Psi(H) \sum |\sigma_1(P)| \leq H\Psi(H)|I| \ll H\Psi(H).$$

As $\#P_n(H, k, m) \ll H^{n-2}$ and there are only $n(n+1)/2$ different pairs (k, m) we obtain the following estimate

$$\sum_{\text{essential intervals } \sigma(P) \text{ with } P \in P_n(H)} |\sigma(P)| \ll H^{n-1}\Psi(H).$$

Thus, by the Borel-Cantelli Lemma and the convergence of (2), the set of points x of $W_{\text{med}}(\Psi) \cap I$ which belong to infinitely many essential intervals is of measure zero.

Now let $\sigma(P)$ be non-essential. Then, by definition and (19) there is a choice of k, m, R such that $P \in P_n(H, k, m, R)$ and there is a $Q \in P_n(H, k, m, R)$ different from P such that

$$\sigma(P) \subset \sigma_1(P) \subset \sigma_2(P) \cap \sigma_2(Q).$$

On the set $\sigma_2(P) \cap \sigma_2(Q)$ both P and Q satisfy (20) and so does the difference $P(x) - Q(x) = b_m x^m + b_k x^k$. It is not difficult to see that $b_m \neq 0$ if H is big enough. Therefore using (20) we get

$$(21) \quad \left| x^{m-k} + \frac{b_k}{b_m} \right| \ll \frac{H^{-1+4\delta}}{|b_m|} \leq H^{-1+4\delta} \quad \text{and} \quad \max\{|b_m|, |b_k|\} \ll H^{2\delta}.$$

Now let x belongs to infinitely many non-essential intervals. Without loss of generality we assume that x is transcendental as otherwise it belongs to a countable set, which is of measure zero. Therefore (21) is satisfied for infinitely many $b_m, b_k \in \mathbb{Z}$. Hence, the inequality

$$\left| x^{m-k} - \frac{p}{q} \right| < q^{-\frac{1-5\delta}{2\delta}}$$

holds for infinitely many $p, q \in \mathbb{Z}$. Taking $\delta = \frac{1}{10}$ so that $\frac{1-5\delta}{2\delta}$ becomes $2 + \delta$ and applying standard Borel-Cantelli arguments (see [8, p. 121]) we complete the proof of the case of a medium derivative for non-essential intervals.

5. The case of a small derivative. In this section we prove that $|W_{\text{small}}(\Psi)| = 0$. We will make use of Theorem 1.4 in [7]. By taking $d = 1$, $\mathbf{f} = (x, x^2, \dots, x^n)$, $U = \mathbb{R}$, $T_1 = \dots = T_n = H$, $\theta = H^{-n+1}$, $K = H^{-\delta}$ in that theorem, we arrive at

THEOREM 2. *Let $x_0 \in \mathbb{R}$ and $\delta' = \frac{\min(\delta, n-1)}{(n+1)(2n-1)}$. Then there exists a finite interval $I_0 \subset \mathbb{R}$ containing x_0 and a constant $E > 0$ such that*

$$\left| \bigcup_{P \in P_n, 0 < H(P) \leq H} \left\{ x \in I_0 : |P(x)| < H^{-n+1}, |P'(x)| < H^{-\delta} \right\} \right| \leq EH^{-\delta'}.$$

In particular Theorem 2 implies that for any $\delta > 0$ the set of $x \in \mathbb{R}$, for which there are infinitely many polynomials $P \in P_n$ satisfying the system

$$(22) \quad |P(x)| < H(P)^{-n+1}, \quad |P'(x)| < H(P)^{-\delta},$$

has zero measure. Indeed, this set consists of points $x \in I_0$ which belong to infinitely many sets

$$\tau_m = \{x \in I_0 : (22) \text{ holds for some } P \in P_n \text{ with } 2^{m-1} < H(P) \leq 2^m\}$$

By Theorem 2, $|\tau_m| \ll 2^{-m\delta'}$ with $\delta' > 0$. Therefore, $\sum_{m=1}^{\infty} |\tau_m| < \infty$ and the Borel-Cantelli Lemma completes the proof of the claim.

Taking into account (6), this completes the proof of the case of a small derivative and the proof of Theorem 1.

6. Concluding remarks. An analogue of Theorem 1 when P is assumed to be irreducible over \mathbb{Q} and primitive (*i.e.* with coprime coefficients) can also be sought. To make it more precise, let $P_n^*(H)$ be the subset of

$P_n(H)$ consisting of primitive irreducible polynomials P of degree $\deg P = n$ and height $H(P) = H$. Now the set of primitive irreducible polynomials of degree n is $P_n^* = \bigcup_{H=1}^{\infty} P_n^*(H)$. Let $W_n^*(\Psi)$ be the set of $x \in \mathbb{R}$ such that there are infinitely many $P \in P_n^*$ satisfying (1).

THEOREM 3. *Let $\Psi : \mathbb{R} \rightarrow \mathbb{R}^+$ be arbitrary function such that the sum*

$$(23) \quad \sum_{H=1}^{\infty} \frac{\#P_n^*(H)}{H} \Psi(H)$$

converges. Then $|W_n^(\Psi)| = 0$.*

For $n = 1$ the proof of Theorem 3 is a straightforward application of the Borel-Cantelli Lemma and we again refer to [8, p.121]. For $n > 1$ the proof follows from the following 2 observations: 1) $W_n^*(\Psi) \subset W_n(\Psi)$ and 2) $\#P_n^*(H) \asymp H^n$. The second one guarantees the converges of (2), which now implies $0 \leq |W_n^*(\Psi)| \leq |W_n(\Psi)| = 0$. The proof of the relation $\#P_n^*(H) \asymp H^n$ is elementary and is left for the reader. In fact, one can easily estimate the number of primitive reducible polynomials in P_n and take them off the set of all primitive polynomials in P_n which is well known to contain at least constant $\times \#P_n(H)$.

The Duffin-Schaeffer conjecture. The conjecture states that for $n = 1$ if (23) diverges then $|\mathbb{R} \setminus W_n^*(\Psi)| = 0$. The multiple $\#P_1^*(H)$ in sum (23) becomes $\asymp \varphi(H)$, where φ is the Euler function.

The following problem can be regarded as the generalization of the Duffin-Schaeffer conjecture for integral polynomials of higher degree:

Prove that $|\mathbb{R} \setminus W_n^(\Psi)| = 0$ whenever (23) diverges.*

Alternatively, for $n > 1$ one might investigate the measure of $\mathbb{R} \setminus W_n(\Psi)$. So far it is unclear if for $n > 1$ $|\mathbb{R} \setminus W_n^*(\Psi)| = 0$ is equivalent to $|\mathbb{R} \setminus W_n(\Psi)| = 0$, which is another intricate question.

A remark on manifolds. In the metric theory of Diophantine approximation on manifolds one usually studies sets of Ψ -approximable points lying on a manifold with respect to the measure induced on that manifold. Mahler's problem and its generalisations can be regarded as Diophantine approximation on the Veronese curve (x, x^2, \dots, x^n) .

A point $\mathbf{f} \in \mathbb{R}^n$ is called Ψ -approximable if

$$\|\mathbf{a} \cdot \mathbf{f}\| < \Psi(|\mathbf{a}|_{\infty}),$$

for infinitely many $\mathbf{a} \in \mathbb{Z}^n$, where $|\mathbf{a}|_\infty = \max_{1 \leq i \leq n} |a_i|$ for $\mathbf{a} = (a_1, \dots, a_n)$, $\|x\| = \min\{|x - z| : z \in \mathbb{Z}\}$ and $\Psi : \mathbb{R} \rightarrow \mathbb{R}^+$.

Let $\mathbf{f} : U \rightarrow \mathbb{R}^n$ be a map defined on an open set $U \subset \mathbb{R}^d$. We say that \mathbf{f} is *non-degenerate at* $\mathbf{x}_0 \in U$ if for some $l \in \mathbb{N}$ the map \mathbf{f} is l times continuously differentiable on a sufficiently small ball centered at \mathbf{x}_0 and there are n linearly independent over \mathbb{R} partial derivatives of \mathbf{f} at \mathbf{x}_0 of orders up to l . We say that \mathbf{f} is *non-degenerate* if it is non-degenerate almost everywhere on U . The non-degeneracy of a manifold is naturally defined via the non-degeneracy of its local parameterisation.

In 1998 D. Kleinbock and D. Margulis proved the Baker-Sprindžuk conjecture by showing that any non-degenerate manifold is strongly extremal. In particular, this implies an analogue of Mahler's problem for non-degenerate manifolds. A few years later an analogue of A. Baker's conjecture with monotonic Ψ (normally called a Groshev type theorem for convergence) has independently been proven by V. Beresnevich [6] and by V. Bernik, D. Kleinbock and G. Margulis [7] for non-degenerate manifolds. It is also remarkable that the proofs were given with different methods. The divergence counterpart (also for monotonic Ψ) has been established in [3]. In [7] a multiplicative version of the Groshev type theorem for convergence has also been given.

Theorem 1 of this paper can be readily generalised for non-degenerate curves: *Given a non-degenerate map $\mathbf{f} : I \rightarrow \mathbb{R}^n$ defined on an interval I , for any function $\Psi : \mathbb{R} \rightarrow \mathbb{R}^+$ such that the sum (2) converges for almost all $x \in I$ the point $\mathbf{f}(x)$ is not Ψ -approximable.* Even further, using the slicing technique of Pyartly [11] one can extend this for a class of n -differentiable non-degenerate manifolds which can be foliated by non-degenerate curves. In particular, this class includes arbitrary non-degenerate analytic manifold. However with the technique in our disposal we are currently unable to prove the following

CONJECTURE. Let $\mathbf{f} : U \rightarrow \mathbb{R}^n$ be a non-degenerate map, where U is an open subset of \mathbb{R}^d . Then for any function $\Psi : \mathbb{R} \rightarrow \mathbb{R}^+$ such that the sum (2) converges for almost all $\mathbf{x} \in U$ the point $\mathbf{f}(\mathbf{x})$ is not Ψ -approximable.

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Institute of Mathematics
Academy of Sciences of Belarus
220072, Surganova 11, Minsk, Belarus
 beresnevich@im.bas-net.by